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Midterm

MATH 534A
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1. Let

$$R(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$

$R(t)$ is an orthogonal matrix. It preserves the lengths of vectors in $\mathbb{R}^3$ and so induces a map $R(t) : S^2 \to S^2$ on the two-dimensional unit sphere. For $p \in S^2$ the smooth curve,

$$t \mapsto c_p(t) = R(t)p = R(t) \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} \in S^2$$

goes through p at $t = 0$ and so defines a tangent vector $V_p = c_p'(0) \in T_pS^2$. Find the coordinate representation for this tangent vector in the u coordinates of stereographic projection from the north pole. Recall that:

$$u(p) = \frac{(p^1, p^2)}{1 - p^3}$$

and

$$p = \left(\frac{2u}{|u|^2 + 1}, \frac{|u|^2 - 1}{|u|^2 + 1} \right), \text{ where } u = (u^1, u^2).$$

That is find $f^j(u)$ for $j = 1, 2$ so that,

$$u_{*p}(V_p) = f^1(u) \frac{\partial}{\partial u^1} + f^2(u) \frac{\partial}{\partial u^2}$$

Solution. Consider R acting on p coordinate-wise and represent this action by $p^i \mapsto Rp^i$. To represent this tangent vector at $t = 0$, consider the following for the first coordinate

$$\begin{aligned} \frac{d}{dt}\left(\frac{Rp^1}{1 - Rp^3} \right)\Bigg|_{t=0} &= \frac{(Rp^1)'(1 - Rp^3) - (1 - Rp^3)'Rp^1}{(1 - Rp^3)^2}\Bigg|_{t=0} \\ &= \frac{(p^2 \cos t - p^3 \sin t)p^1}{(1 - (p^2 \sin t + p^3 \cos t))^2}\Bigg|_{t=0} \\ &= \frac{-p^1 p^2}{(1 - p^3)^2} \end{aligned}$$

and in the second coordinate

$$\begin{aligned}
\frac{d}{dt}\left(\frac{Rp^2}{1-Rp^3}\right)\Bigg|_{t=0} &= \frac{(Rp^2)'(1-Rp^3)-(1-Rp^3)'Rp^2}{(1-Rp^3)^2}\Bigg|_{t=0} \\
&= \frac{(-p^2\sin t-p^3\cos t)(1-(p^2\sin t+p^3\cos t))+(p^2\cos t-p^3\sin t)^2}{(1-(p^2\sin t+p^3\cos t))^2}\Bigg|_{t=0} \\
&= \frac{-p^3(1-p^3)+(p^2)^2}{(1-p^3)^2}.
\end{aligned}$$

So, we have the following in terms of $u(p)$,

$$\begin{aligned}
f^1(u(p)) &= \frac{-p^1p^2}{(1-p^3)^2} \\
f^2(u(p)) &= \frac{-p^3(1-p^3)+(p^2)^2}{(1-p^3)^2}.
\end{aligned}$$

Now apply the coordinate transformation for stereographic projection from the north pole to get

$$\begin{aligned}
f^1(u) &= \frac{\frac{-4u^1u^2}{(|u|^2+1)^2}}{\left(1-\frac{|u|^2-1}{|u|^2+1}\right)^2} \\
&= \frac{-4u^1u^2}{(|u|^2+1-(|u|^2-1))^2} \\
&= -u^1u^2 \\
f^2(u) &= \frac{\left(\frac{1-|u|^2}{|u|^2+1}\right)\left(1-\frac{|u|^2-1}{|u|^2+1}\right)+\left(\frac{2u_2}{|u|^2+1}\right)^2}{\left(1-\frac{|u|^2-1}{|u|^2+1}\right)^2} \\
&= \frac{(1-|u|^2)(|u|^2+1-(|u|^2-1))+4(u^2)^2}{(|u|^2+1-(|u|^2-1))^2} \\
&= \frac{2(1-|u|^2)+4(u^2)^2}{4} \\
&= \frac{1}{2}(1-|u|^2)+(u^2)^2
\end{aligned}$$

Therefore, we have the coordinate representation for $V_p \in T_pS^2$,

$$\begin{aligned}
u_{*p}(V_p) &= f^1(u)\frac{\partial}{\partial u^1}+f^2(u)\frac{\partial}{\partial u^2} \\
&= -u^1u^2\frac{\partial}{\partial u^1}+\frac{1}{2}(1-|u|^2)+(u^2)^2\frac{\partial}{\partial u^2}
\end{aligned}$$

$\square$

2. Suppose that $g : \mathbb{R}^n \to \mathbb{R}$ is a smooth function and 0 is a regular value so that $M = g^{-1}(0)$ is a regular submanifold. Suppose that U is an open neighborhood of M in $\mathbb{R}^n$ and $F : U \to \mathbb{R}$ is a smooth function on U, and write $f = F|_M$ for the restriction of F to M. Show that for $p \in M$,

$$F_{*,p}(v) = f_{*,p}(v) \text{ for } v \in T_pM$$

Use this to show that p is a critical point for f on M if and only if there exists a constant λ so that,

$$F_{*,p} = \lambda g_{*,p}$$

The constant λ is called a Lagrange multiplier. Use this to find the critical points for the restriction of the function $F(x, y, z) = xy$ to the unit sphere, $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. Hint: a linear functional on a finite dimensional vector space is determined up to a constant multiple by its null space.

Solution. Represent v by the derivative of the germ of a $C_p^\infty(\mathbb{R})$ function evaluated at $t = 0$; call it $c(t) : \mathbb{R} \to \mathbb{R}$. Therefore $v = c'(0)$, and hence it follows that

$$\begin{aligned} F_{*p}(v) &= F_{*p}(c'(0)) \\ &= f_{*p}(c'(0)) \\ &= f_{*p}(v) \end{aligned}$$

□

Now, consider $M = g^{-1}(0)$. It follows that M is a level set for g and that, for any $c \in C_p^\infty(\mathbb{R})$, $g \circ c(t)$ is constant for all $t \in \mathbb{R}$. Therefore, $\frac{d}{dt}(g \circ c(t)) = 0$ for all t. This implies that $\text{rk}(g_{*p}) = 1$ and $\dim(N(g_{*p})) = n - 1$, for M n-dimensional. Now consider $f_{*,p}$ when p is critical. Then

$$f_{*p} = 0 = F_{*p}$$

Now, because $F_{*p} = 0$, $\dim(N(F_{*p})) = \dim(N(g_{*p})) = n - 1$. We can apply the hint given in the problem statement to conclude

$$F_{*p} = \lambda g_{*p}$$

for some $\lambda \in \mathbb{R}$ when p is critical.

The notion of p being a critical point is crucial. If p were a regular point, then $F_{*p} \neq 0$. From the argument above, $g_{*p} = 0$, and so there could not exist a constant λ s.t. $F_{*p} = \lambda g_{*p}$. So, if $F_{*p} = \lambda g_{*p}$, p must be a critical point.

Therefore, p is a critical point for f on M if and only if there exists a constant λ so that,

$$F_{*,p} = \lambda g_{*,p}$$

□

We can use the above construction to find the critical points of $F(x,y,z) = xy$ restricted to the unit sphere $g(x,y,z) = x^2+y^2+z^2-1 = 0$. Notice $S^2 = g^{-1}(\{0\})$. Calculate the derivations at p,

$$\begin{aligned} g_{*p} &= 2xdx_p + 2ydy_p + 2zdz_p \\ F_{*p} &= ydx_p + xdy_p. \end{aligned}$$

We are looking for points at which

$$F_{*p} = \lambda g_{*p}$$

which would imply

$$ydx_p + xdy_p = \lambda(2xdx_p + 2ydy_p + 2zdz_p).$$

Clearly, z must be 0 for equality to hold. Additionally, $x = \pm y$ to ensure proper cancellation. This then shows that $\lambda = \pm\frac{1}{2}$. Therefore, the critical points on S^2 are $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right)$. □

3. Show that -1 is a regular value for the function,

$$F(x,y,z) = z^2 - x^2 - y^2$$

So that $M = F^{-1}(-1)$ is a regular submanifold of $\mathbb{R}^3$. Show that the map,

$$S^1 \times \mathbb{R} \ni (p,z) \to \phi(p,z) = (\sqrt{z^2+1}p, z) \in M$$

is a diffeomorphism of $S^1 \times \mathbb{R}$ with M.

Solution. Consider the partial derivatives

$$\frac{\partial F}{\partial x} = -2x,\ \frac{\partial F}{\partial y} = -2y,\ \frac{\partial F}{\partial z} = 2z.$$

These partials all vanish at $(0,0,0)$ only and $(0,0,0) \notin F^{-1}(\{-1\})$. So, we can conclude that -1 is a regular value for F. □

To show that ϕ as defined is a diffeomorphism, it must be a bijection, C^∞, and have a C^∞ inverse. To show that ϕ is a bijection, it is sufficient to show that $\phi^{-1} : M \to S^1 \times \mathbb{R}$ is a function. Observe that

$$\phi^{-1}(x,y,z) = \left(\frac{x}{\sqrt{z^2+1}}, \frac{y}{\sqrt{z^2+1}}, z\right).$$

This map is a function since each point in the domain M maps to a single point in $S^1 \times \mathbb{R}$. By inspection, ϕ is C^∞ since there is no problem with partial differentiation being infinite in any of the coordinates. Now, to see if ϕ^{-1} is C^∞, check the determinant of the Jacobian matrix

$$\det D\phi = \begin{vmatrix} \sqrt{z^2+1} & 0 & \frac{xz}{\sqrt{z^2+1}} \\ 0 & \sqrt{z^2+1} & \frac{yz}{\sqrt{z^2+1}} \\ 0 & 0 & 1 \end{vmatrix} = z^2+1$$

Since this determinant is non-zero everywhere, ϕ is locally invertible everywhere on $S^1 \times \mathbb{R}$ by the Inverse Function Thm. It follows that ϕ^{-1} is C^∞. So, ϕ is diffeomorphism. □